MATH 251, Mariano Echeverria

Problems on Scalar Fields

This material corresponds roughly to sections 14.1, 14.2, 14.3, 14.4, 14.5 and 14.6 in the book.

Problem 1. [Maxwell relation] Thermodynamics teaches that the energy E of a rigid container of gas is a function of its entropy S and volume V: $E = E(S, V)$. Its temperature is given by $T = \frac{\partial E}{\partial S}$ and its pressure by $P = -\frac{\partial E}{\partial V}$. Show that $\frac{\partial T}{\partial V} = -\frac{\partial P}{\partial S}$. This is a Maxwell relation.

The relation is usually written $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$ in physics and chemistry texts to make it clear that S is held constant on the left and V is held constant on the right when computing the partial derivatives.

Notice that

$$\begin{aligned} &\frac{\partial T}{\partial V} \\ =&\frac{\partial}{\partial V}\left(\frac{\partial E}{\partial S}\right) \\ =&\frac{\partial}{\partial S}\left(\frac{\partial E}{\partial V}\right) \\ =&-\frac{\partial P}{\partial S} \end{aligned}$$

Problem 2. [Wave equation] Let $u(x, t) = f(x - vt) + g(x + vt)$ where f, g are scalar valued functions (so they take real values). Show that

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

by applying the chain rule (or tree diagrams). This is a *partial differential equation,* called the *wave equation.*

We define the variables

$$\begin{cases} z = x - vt \\ w = x + vt \end{cases} \tag{2}$$

so that we can write the function u as

$$u = f(z) + g(w) \tag{3}$$

So using the tree diagram we can consider u as a function of z, w, each of which is a function of x, t:

$$\begin{array}{ccccccc} & & & u & & & \\ & & / & & \backslash & & \\ & z & & & & w & \\ & /\backslash & & & & /\backslash & \\ x & & t & & x & & t \end{array} \tag{4}$$

Using the chain rule:

$$\begin{aligned} u_x =& u_z z_x + u_w w_x \\ u_t =& u_z z_t + u_w w_t \end{aligned}$$

$$\begin{aligned} u_{xx} =& (u_z)_x z_x + u_z z_{xx} + (u_w)_x w_x + u_w w_{xx} \\ =& (u_{zz} z_x + u_{zw} w_x) z_x + u_z z_{xx} + (u_{wz} z_x + u_{ww} w_x) w_x + u_w w_{xx} \\ =& (\frac{d^2 f}{dz^2} \cdot 1 + 0 \cdot 1)1 + \frac{df}{dz} \cdot 0 + (0 \cdot 1 + \frac{d^2 g}{dw^2} \cdot 1)1 + \frac{dg}{dw} \cdot 0 \\ =& \frac{d^2 f}{dz^2} + \frac{d^2 g}{dw^2} \end{aligned}$$

$$\begin{aligned} u_{tt} =& (u_z)_t z_t + u_z z_{tt} + (u_w)_t w_t + u_w w_{tt} \\ =& (u_{zz} z_t + u_{zw} w_t) z_t + u_z z_{tt} + (u_{wz} z_t + u_{ww} w_t) w_t + u_w w_{tt} \\ =& (\frac{d^2 f}{dz^2} \cdot (-v) + 0 \cdot (v))(-v) + \frac{df}{dz} \cdot 0 + (0 \cdot (-v) + \frac{d^2 g}{dw^2} \cdot (v))(v) + \frac{dg}{dw} \cdot 0 \\ =& v^2 \frac{d^2 f}{dz^2} + v^2 \frac{d^2 g}{dw^2} \end{aligned}$$

Notice that the formulas for u_{xx} and u_{tt} clearly show that

$$v^2 u_{xx} = u_{tt} \tag{5}$$

so u satisfies the wave equation.

Problem 3. The equation $z = xf(y/x)$ defines a surface whenever $x \neq 0$ and f is a real valued function. Find the equation of the tangent plane to the surface passing through the point $(x_0, y_0, x_0 f(y_0/x_0))$. Does the origin $(0, 0, 0)$ belong to this plane?

In this case the equation of the surface is

$$g(x, y, z) = xf(y/x) - z \tag{6}$$

The gradient of g is

$$\begin{aligned} \nabla g &= \big(f(y/x) + xf'(y/x) \cdot (-y/x^2), xf'(y/x) \cdot (1/x), -1\big) \\ &= \Big(f(y/x) - \frac{y}{x} f'(y/x), f'(y/x), -1\Big) \end{aligned}$$

At the point $(x_0, y_0, x_0 f(y_0/x_0))$ the normal vector will be

$$\mathbf{n} = \nabla g(x_0, y_0, z_0) = \left(f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0), f'(y_0/x_0), -1 \right) \tag{7}$$

so the equation of the normal plane to the surface passing through the point is

$$\left[f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0) \right] x + f'(y_0/x_0) y - z = \left[f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0) \right] x_0 + f'(y_0/x_0) \cdot y_0 - x_0 f(y_0/x_0) \tag{8}$$

which we can simply as

$$\boxed{\left[f(y_0/x_0) - \frac{y_0}{x_0} f'(y_0/x_0) \right] x + f'(y_0/x_0) y - z = 0} \tag{9}$$

Observe that the origin $(0, 0, 0)$ does satisfy the equation of this plane.

Problem 4. Let $f(x, y) = x - y^2$ and $g(x, y) = 2x + \ln y$. Show that the level curves of f and g are orthogonal at every point where they meet.

The level curve at height $z = c$ for f solves the equation

$$x - y^2 = c \tag{10}$$

This can be used to write x as a function of y as

$$x = c + y^2 \tag{11}$$

so the position vector of this level curve is

$$\mathbf{r}_f(y) = (x, y, z) = (c + y^2, y, c) \tag{12}$$

so the velocity vector to the curve is

$$\boxed{\frac{d\mathbf{r}_f}{dy} = (2y, 1, 0)} \tag{13}$$

On the other hand, the level curve at height $z = c$ for g solves the equation

$$2x + \ln y = c \tag{14}$$

This can be used to write x as a function of y as

$$x = \frac{c - \ln y}{2} \tag{15}$$

so the position vector of this level curve is

$$\mathbf{r}_g(y) = (x, y, z) = \left(\frac{c - \ln y}{2}, y, c \right) \tag{16}$$

so the velocity vector to the curve is

$$\boxed{\frac{d\mathbf{r}_g}{dy} = \left(-\frac{1}{2y}, 1, 0\right)} \tag{17}$$

Notice that

$$\frac{d\mathbf{r}_f}{dy} \cdot \frac{d\mathbf{r}_g}{dy} = (2y, 1, 0) \cdot \left(-\frac{1}{2y}, 1, 0\right) = -1 + 1 = 0 \tag{18}$$

so in fact the velocity vectors are orthogonal.

Problem 5. If three resistors R_1, R_2, R_3 are connected in parallel, the total electrical resistance is determined by the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{19}$$

Find $\frac{\partial R}{\partial R_1}$.

We differentiate both sides of the equation with respect to R_1:

$$\frac{\partial}{\partial R_1}\left(R^{-1}\right) = \frac{\partial}{\partial R_1}\left(\frac{1}{R_1}\right) + \frac{\partial}{\partial R_1}\left(\frac{1}{R_2}\right) + \frac{\partial}{\partial R_1}\left(\frac{1}{R_3}\right) \tag{20}$$

By the chain rule this is the same as

$$-\left(\frac{1}{R}\right)^2 \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2} \tag{21}$$

so this is the same as

$$-\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2 \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2} \tag{22}$$

In other words

$$\boxed{\frac{\partial R}{\partial R_1} = \frac{1}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2} \cdot \frac{1}{R_1^2} = \frac{1}{\left(\frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}\right)^2} \frac{1}{R_1^2} = \frac{R_2^2R_3^2}{(R_2R_3 + R_1R_3 + R_1R_2)^2}} \tag{23}$$

Another way to find is by finding R as a function of R_1, R_2, R_3 explicitly. Namely, notice that the equation for $\frac{1}{R}$ is equivalent to

$$\frac{1}{R} = \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3} \tag{24}$$

or

$$R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2} \tag{25}$$

We can find the partial derivative using the quotient rule

$$\frac{\partial R}{\partial R_1} = \frac{R_2R_3(R_2R_3 + R_1R_3 + R_1R_2) - R_1R_2R_3(R_3 + R_2)}{(R_2R_3 + R_1R_3 + R_1R_2)^2} = \frac{R_2^2R_3^2}{(R_2R_3 + R_1R_3 + R_1R_2)^2} \tag{26}$$

Problem 6. Suppose that a duck is swimming in a circle, $x = \cos t$, $y = \sin t$, while the water temperature is given by the formula $T = x^2e^y - xy^3$. Find $\frac{dT}{dt}$ using the chain rule.

Here is the diagram

$$\begin{array}{ccccc} & & T & & \\ & / & & \backslash & \\ x & & & & y \\ | & & & & | \\ t & & & & t \end{array} \tag{27}$$

Therefore

$$\begin{aligned} &\frac{dT}{dt} \\ =&\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dx}{dt} \\ =&(2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)(\cos t) \\ =&(2\cos te^{\sin t} - \sin^3 t)(-\sin t) + (\cos^2 te^{\sin t} - 3\cos t\sin^2 t)(\cos t) \end{aligned}$$

Problem 7. Show that the tangent plane at each point (x_0, y_0, z_0) of the cone $z = \sqrt{x^2 + y^2}$, (with $x_0 \neq 0$, $y_0 \neq 0$) contains the line passing through (x_0, y_0, z_0) and the origin.

Notice that the equation of the cone can be written as $z^2 = x^2 + y^2$ so we can work with the equation of the surface

$$g(x, y, z) = x^2 + y^2 - z^2 \tag{28}$$

The gradient is

$$\nabla g = (2x, 2y, -2z) \tag{29}$$

So at the point $(x_0, y_0, \sqrt{x_0^2 + y_0^2})$ the equation of the tangent plane is [$\mathbf{n} = (2x_0, 2y_0, -2\sqrt{x_0^2 + y_0^2})$ in this case]

$$\boxed{2x_0x + 2y_0y - 2\sqrt{x_0^2 + y_0^2}z = 2x_0x_0 + 2y_0y_0 - 2(x_0^2 + y_0^2) = 0} \tag{30}$$

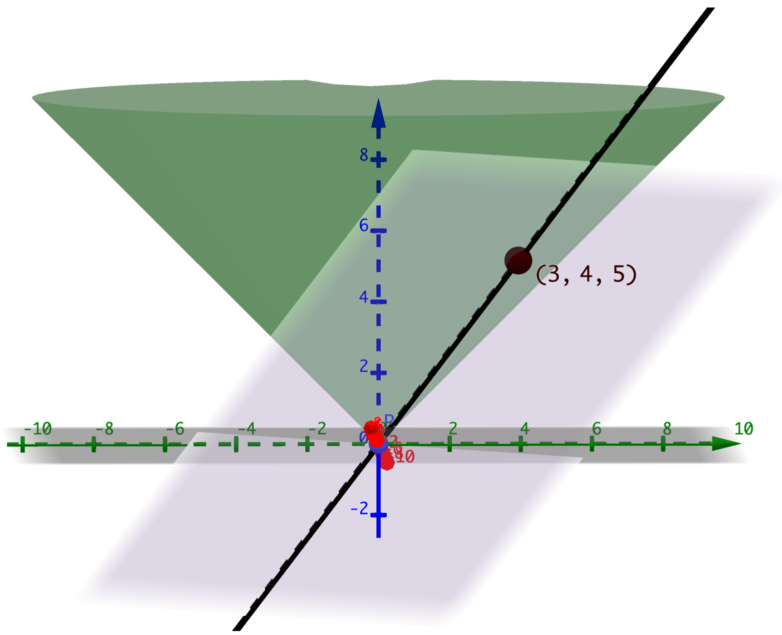
The line passing through $(x_0, y_0, z_0) = (x_0, y_0, \sqrt{x_0^2 + y_0^2})$ and the origin is

$$\mathbf{r}(t) = t(x_0, y_0, \sqrt{x_0^2 + y_0^2}) \tag{31}$$

so we need to check that each point on the line satisfies the equation of the plane. We plug in $tx_0, ty_0, t\sqrt{x_0^2 + y_0^2}$ as our values for x, y, z on the left hand side of the equation of the plane

$$
\begin{aligned}
&2x_0x + 2y_0y - z \\
=&2x_0tx_0 + 2y_0ty_0 - 2\sqrt{x_0^2+y_0^2}t\sqrt{x_0^2+y_0^2} \\
=&t\left(2x_0^2 + 2y_0^2 - 2(x_0^2+y_0^2)\right) \\
=&t \cdot 0 \\
=&0
\end{aligned}
$$

Notice that this equation holds regardless of the specific value of t, so we verified the assumption.



Problem 8. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$.

a) Show that $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$ whenever $r \neq 0$.

Notice that x, y, z appear in a symmetric fashion in each of the formulas, so it suffices to compute $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$ to figure out the pattern. For this we use the product rule

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial}{\partial x}r = -\frac{1}{r^2}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = -\frac{1}{x^2+y^2+z^2}\cdot\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{x}{r^3} \tag{32}$$

Therefore

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial z}r^{-1}\right) = \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right) = -\frac{1}{r^3}\mathbf{r} \tag{33}$$

b) What is $\|\nabla\left(\frac{1}{r}\right)\|$?

$$\|\nabla r^{-1}\| = \|-\frac{1}{r^3}\mathbf{r}\| = \frac{1}{r^3}\|\mathbf{r}\| = \frac{r}{r^3} = \frac{1}{r^2} \tag{34}$$

c) In electrostatics, the force $\mathbf{F}_e$ of attraction between two particles of opposite charge is given by $\mathbf{F}_e = k\frac{\mathbf{r}}{r^3}$. A potential function V for the electrostatic force is a scalar function $V = V(x, y, z)$ such that $\nabla V = -\mathbf{F}_e$ (here I am using the physicist convention for the potential). Find a potential V for $\mathbf{F}_e$.

Take

$$V = -\frac{k}{r} \tag{35}$$

and notice that thanks to part a)

$$\nabla V = -\nabla(k/r) = -k\nabla(1/r) = k\frac{\mathbf{r}}{r^3} = \mathbf{F}_e \tag{36}$$

as desired.

Problem 9. Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any member of the family of surfaces $x^2 + 1 = (2 - 4a)y^2 + az^2$ at the point of intersection $(1, -1, 2)$.

The equation for the first surface is

$$g_1(x, y, z) = x^2 - 2yz + y^3 - 4 = 0 \tag{37}$$

Which has gradient

$$\nabla g_1 = (2x, -2z + 3y^2, -2y) \tag{38}$$

so the normal vector at the point $(1, -1, 2)$ is

$$\nabla g_1(1, -1, 2) = (2, -1, 2) \tag{39}$$

The equation for the second surface is

$$g_2(x, y, z) = x^2 + 1 - (2 - 4a)y^2 - az^2 \tag{40}$$

so the gradient is

$$\nabla g_2 = (2x, -2(2 - 4a)y, -2az) \tag{41}$$

The normal vector at the point $(1, -1, 2)$ is

$$\nabla g_2(1, -1, 2) = (2, 4 - 8a, -4a) \tag{42}$$

Notice that $\nabla g_1(1, -1, 2)$ and $\nabla g_2(1, -1, 2)$ are orthogonal since

$$(2, -1, 2) \cdot (2, 4 - 8a, -4a) = 4 + 8a - 4 - 8a = 0 \tag{43}$$

so the tangent planes indeed intersect orthogonally.

Problem 10. Find the directional derivative of $U(x, y, z) = 2x^3y - 3y^2z$ at the point $P = (1, 2, -1)$ in a direction toward the point $Q = (3, -1, 5)$.

The direction vector is

$$\mathbf{v} = \overrightarrow{PQ} = Q - P = (3, -1, 5) - (1, 2, -1) = (2, -3, 6) \tag{44}$$

We need to normalize it

$$\mathbf{e}_\mathbf{v} = \frac{(2, -3, 6)}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{1}{7}(2, -3, 6) \tag{45}$$

On the other hand the, gradient is

$$\nabla U = (6x^2y, 2x^3 - 6yz, -3y^2) \tag{46}$$

so at the point $(1, 2, -1)$ the gradient is

$$\nabla U(1, 2, -1) = (12, 14, -12) \tag{47}$$

and the directional derivative is

$$D_{\mathbf{v}}U(1, 2, -1) = \nabla U(1, 2, -1) \cdot \mathbf{e}_{\mathbf{v}} = (12, 14, -12) \cdot \frac{1}{7}(2, -3, 6) = \frac{24 - 42 - 72}{7} = -\frac{90}{7} \tag{48}$$